Finding Mixed-strategy Nash Equilibria in 2 × 2 Games

Introduction

We'll now see explicitly how to find the set of (mixed-strategy) Nash equilibria for two-player games where each player has a strategy space containing two actions (i.e. a "2 × 2 matrix game"). After setting up the analytical framework and deriving some general results for such games, we will apply this technique to two particular games. The first game is a typical and straightforwardly solved example; the second is *nongeneric* in the sense that it has an infinite number of equilibria.[1] For each game we will compute the graph of each player's best-response correspondence and identify the set of Nash equilibria by finding the intersection of these two graphs.

The canonical game

We consider the two-player strategic-form game in Figure 1. We assign rows to player A and columns to player B. A's strategy space is $S_A = \{U, D\}$ and B's is $S_B = \{l, r\}$. Because each player has only two actions, each of her mixed strategies can be described by a single number (p for A and q for B)

© 1992 by Jim Ratliff , <jim@virtualperfection.com>, <http://virtualperfection.com/gametheory>.

1 Here *nongeneric* means that the phenomenon depends very sensitively on the exact payoffs. If the payoffs were perturbed the slightest bit, then the phenomenon would disappear.

belonging to the unit interval [0, 1]. A mixed-strategy profile for this game, then, is an ordered pair $(p,q)\in[0,1]\times[0,1]$. We denote the players' payoffs resulting from pure-strategy profiles by subscripted a's and b's, respectively. E.g. the payoff for A when A plays D and B plays r is a_{Dr}.

		B	
		l: q	r: $1-q$
A	U: p	a_{Ul}, b_{Ul}	a_{Ur}, b_{Ur}
	D: $1-p$	a_{Dl}, b_{Dl}	a_{Dr}, b_{Dr}

Figure 1: The canonical two-player, two-action-per-player strategic-form game.

The pure-strategy equilibria, if any, of such a game are easily found by inspection of the payoffs in each cell, each cell corresponding to a pure-strategy profile. A particular pure-strategy profile is a Nash equilibrium if and only if 1 that cell's payoff to the row player (viz. A) is a (weak) maximum over all of A's payoffs in that column (otherwise the row player could profitably deviate by picking a different row given B's choice of column) and 2 that cell's payoff to the column player (viz. B) is a (weak) maximum over all of B's payoffs in that row. For example, the pure-strategy profile (U,r) would be a Nash equilibrium if and only if the payoffs were such that $a_{Ur}\geq a_{Dr}$ and $b_{Ur}\geq b_{Ul}$.

Best-response correspondences

Finding the pure-strategy equilibria was immediate. Finding the mixed-strategy equilibria takes a little more work, however. To do this we need first to find each player's best-response correspondence. We will show in detail how to compute player A's correspondence. Player B's is found in exactly the same way.

Player A's best-response correspondence specifies, for each mixed strategy q played by B, the *set* of mixed strategies p which are best responses for A. I.e. it is a correspondence p^* which associates with every $q\in[0,1]$ a set $p^*(q)\subset[0,1]$ such that every element of $p^*(q)$ is a best response by A to B's choice q. The *graph* of p^* is the set of points

$$\{(p,q)\colon q\in[0,1],\ p\in p^*(q)\}. \tag{1}$$

A's payoff as a function of the mixed-strategy profile

To find A's best-response correspondence we first compute her expected payoff for an arbitrary mixed-strategy profile (p,q) by weighting each of A's pure-strategy profile payoffs by the probability of that profile's occurrence as determined by the mixed-strategy profile (p,q):[1]

$$u_A(p;\,q)=pqa_{Ul}+p(1-q)a_{Ur}+(1-p)qa_{Dl}+(1-p)(1-q)a_{Dr}. \tag{2}$$

A's utility maximization problem is

---

1 The semicolon in "$u(p;\,q)$" is used to denote that, while p is a choice variable for A, q is a parameter outside of A's control.

$$\max_{p\in[0,1]} u_A(p;\, q). \tag{3}$$

Because p is A's choice variable, it will be convenient to rewrite equation (2) as an affine function of p:[1]

$$u_A(p;q) \;=\; p[(a_{Ul}-a_{Ur}-a_{Dl}+a_{Dr})q+(a_{Ur}-a_{Dr})] \;+\; [(a_{Dl}-a_{Dr})q+a_{Dr}], \tag{4a}$$

$$=\delta(q)p+[(a_{Dl}-a_{Dr})q+a_{Dr}]. \tag{4b}$$

Of interest here is the sign of the coefficient of p,

$$\delta(q) \equiv (a_{Ul}-a_{Ur}-a_{Dl}+a_{Dr})q+(a_{Ur}-a_{Dr}), \tag{5}$$

which is itself an affine function of q.

A's best-response correspondence

For a given q, the function $u_A(p;\, q)$ will be maximized with respect to p either 1 at the unit interval's right endpoint (viz. $p=1$) if $\delta(q)$ is positive, 2 at the interval's left endpoint (viz. $p=0$) if $\delta(q)$ is negative, or 3 for every $p\in[0,1]$ if $\delta(q)$ is zero, because $u_A(p;\, q)$ is then constant with respect to p.

Now we consider the behavior of A's best response as a function of q. There are three major cases to consider.

Case 1: complete indifference

A's payoffs could be such that $\delta(q)=0$ for all q.[2] In this case A's best-response correspondence would be independent of q and would simply be the unit interval itself: $\forall q\in[0,1],\; p^*(q)=[0,1]$. In other words A would be willing to play any mixed strategy regardless of B's choice of strategy. The graph of p^* in this case is the entire unit square $[0,1]\times[0,1]$. (See Figure 2.)

1 I would be happy to say *linear* here instead of *affine*. The strict definition of linear seems to be made consistently in linear algebra, but the less restrictive definition seems to be tolerated in other contexts.

2 This would require that in each column A receives the same payoff in each of the two rows.

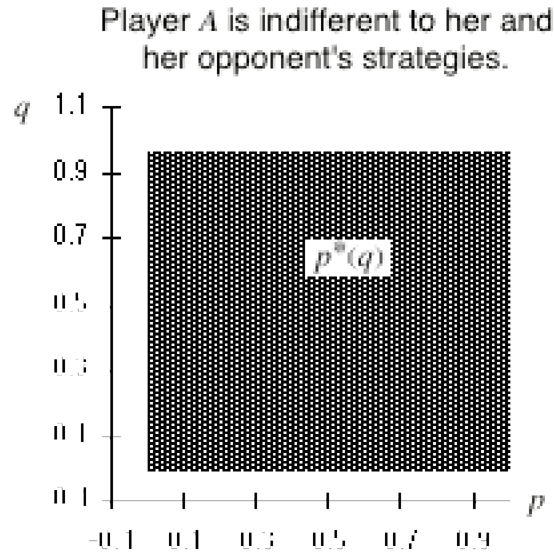

Figure 2: Best-response correspondence when A is completely indifferent.

Case 2: *A* has a dominant pure strategy

If this is not the case, i.e. if $\delta(q)$ is not identically zero, then—because $\delta(q)$ is affine—there will be exactly one value $q^\dagger$ at which $\delta(q^\dagger)=0$. For all q to one side of $q^\dagger$, $\delta(q)$ will be positive; for all q on the other side of $q^\dagger$, $\delta(q)$ will be negative. However, this $q^\dagger$ need not be an element of $[0,1]$. If $q^\dagger \notin [0,1]$, then all $q \in [0,1]$ will lie on a common side of $q^\dagger$ and therefore $\delta(q)$ will have a single sign throughout the interval $[0,1]$. Therefore A will have the same best response for every q (viz. $p=1$ if $\delta(q)>0$ on $[0,1]$; $p=0$, if $\delta(q)<0$ on $[0,1]$); i.e. A has a strongly dominant pure strategy. (See Figure 3.)

Case 3: *A* plays strategically

Now we consider the case where A plays strategically: her optimal strategy depends upon her opponent's strategy. If $q^\dagger \in (0,1)$, then the unit interval is divided into two subintervals—those points to the right of $q^\dagger$ and those points to the left of $q^\dagger$—in each of which A plays a different pure strategy. At exactly $q^\dagger$, A can mix with any $p \in [0,1]$. (See Figure 4.)

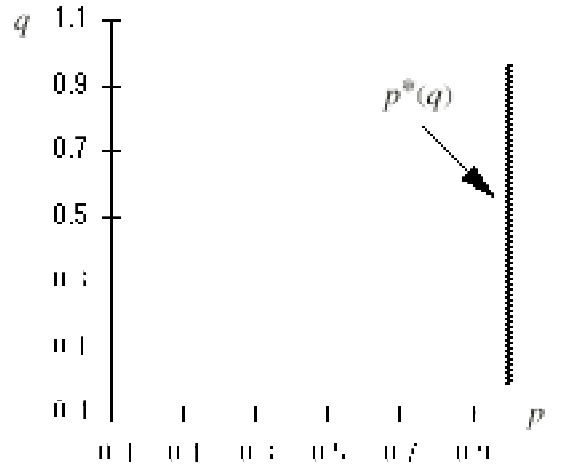

Figure 3: Best-response correspondence when A has a dominant pure strategy ($p=1$).

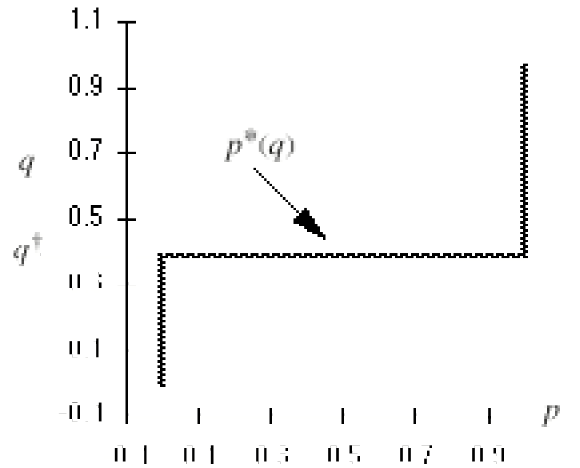

Figure 4: Best-response correspondence when A plays strategically.

If $q^\dagger=0$ or $q^\dagger=1$, then A can mix at $q^\dagger$ and will play a common pure strategy for every other $q\in[0,1]$. In this case the pure strategy which A plays against $q\in(0,1)$ is a weakly dominant strategy. (It is strictly best against $q\in[0,1]-\{q^\dagger\}$ and as good as anything else against $q^\dagger$.)

Summary of the best-response correspondence

We see that—in a 2×2 matrix game—each player will either 1 be free to mix regardless of her opponent's strategy (a case which 'almost never' occurs), 2 play a dominant pure strategy, or 3 be free to mix in response to exactly one of her opponent's possible choices of strategy but for every other choice will play a pure strategy.

B's best-response correspondence

Player B's best-response correspondence specifies, for each mixed strategy p played by A, the set of mixed strategies q which are best responses for B. I.e. it is a correspondence q^* which associates with every $p \in [0, 1]$ a set $q^*(p) \subset [0, 1]$ such that every element of $q^*(p)$ is a best response by B to A's choice p. The graph of q^* is the set of points

$$\{(p, q): p \in [0, 1], \Box\; q \in q^*(p)\}. \tag{6}$$

This correspondence is found using the same method of analysis we used for A's. You will easily show that

$$u_B(q; p) = \gamma(p)q + [(b_{Ur} - b_{Dr})p + b_{Dr}], \tag{7}$$

where

$$\gamma(p) \equiv (b_{Ul} - b_{Ur} - b_{Dl} + b_{Dr})p + (b_{Dl} - b_{Dr}). \tag{8}$$

The Nash equilibria are the points in the intersection of the graphs of A's and B's best-response correspondences

We know that a mixed-strategy profile (p, q) is a Nash equilibrium if and only if 1 p is a best response by A to B's choice q and 2 q is a best response by B to A's choice p. We see from (1) that the first requirement is equivalent to (p, q) belonging to the graph of p^*, and from (6) we see that the second requirement is equivalent to (p, q) belonging to the graph of q^*. Therefore we see that (p, q) is a Nash equilibrium if and only if it belongs to the intersection of the graphs of the best-response correspondences p^* and q^*. We can write the set of Nash equilibria, then, as

$$\{(p, q) \in [0, 1] \times [0, 1]: \; p \in p^*(q), \; q \in q^*(p)\}. \tag{9}$$

A typical example

Consider the 2×2 game in Figure 5. First we immediately observe that there are two pure-strategy Nash equilibria: (U, r) and (D, l).

	l: q	r: $1-q$
U: p	1, −1	3, 0
D: $1-p$	4, 2	0, −1

Figure 5: A typical 2×2 game.

A's best-response correspondence

Now we find A's best-response correspondence. From (5) we see that

$$\delta(q) = -6q + 3, \tag{10}$$

which vanishes at $q^\dagger = {}^1\!/_2$. Because $\delta(q)$ is decreasing in q we see that A will choose the pure strategy $p = 1$ against q's on the interval $[0, {}^1\!/_2)$ and the pure strategy $p = 0$ against q's on the interval $({}^1\!/_2, 1]$. Against $q = q^\dagger = {}^1\!/_2$, A is free to choose any mixing probability. Player A's best-response correspondence p^* is plotted in Figure 6.

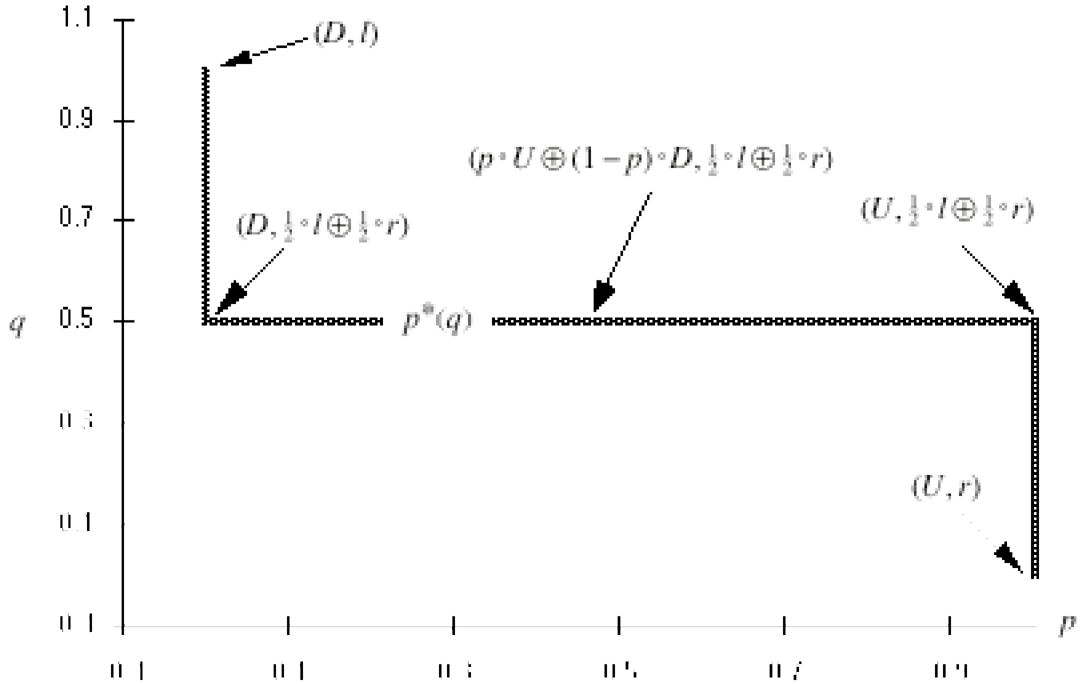

Figure 6: A's best-response correspondence for the game of Figure 5.

We can also derive A's best-response correspondence graphically by plotting her payoff to her different pure strategies as a function of B's mixed-strategy choice q. Using (4b) and (10) we have

$$u_A(p;\, q) = (-6q+3)p + 4q. \tag{11}$$

Evaluating this payoff function at A's pure-strategy choices $p=1$ and $p=0$, respectively, we have

$$u_A(U;\, q) = 3-2q, \tag{12}$$

$$u_A(D;\, q) = 4q. \tag{13}$$

Both of these functions at plotted for $q \in [0,1]$ in Figure 7. These two lines intersect when $q = q^\dagger = {}^1\!/_2$; i.e. they intersect at the mixed strategy for B at which we earlier determined A would be willing to mix. To the left of this point, A's payoff to U is higher than her payoff to D; the reverse is true on the other side of $q^\dagger$. Therefore A's best response is to play U against $q \in [0, {}^1\!/_2)$ and D against $q \in ({}^1\!/_2, 1]$. At the intersection point $q^\dagger$, A is indifferent to playing U or D, so she is free to mix between them. This is exactly the best-response correspondence we derived analytically above.

B's best-response correspondence

We similarly find B's best-response correspondence. From (8) we find that

$$\gamma(p) = -4p+3, \tag{14}$$

which decreases in p and vanishes at $p^\dagger = {}^3\!/_4$. Player B, then, chooses the pure strategy $q=1$ against p's on the interval $[0, {}^3\!/_4)$ and the pure strategy $q=0$ against p's on the interval $({}^3\!/_4, 1]$. Against $p = p^\dagger = {}^3\!/_4$,

B is free to choose any mixing probability. Player B's best-response correspondence q^* is plotted in Figure 8.

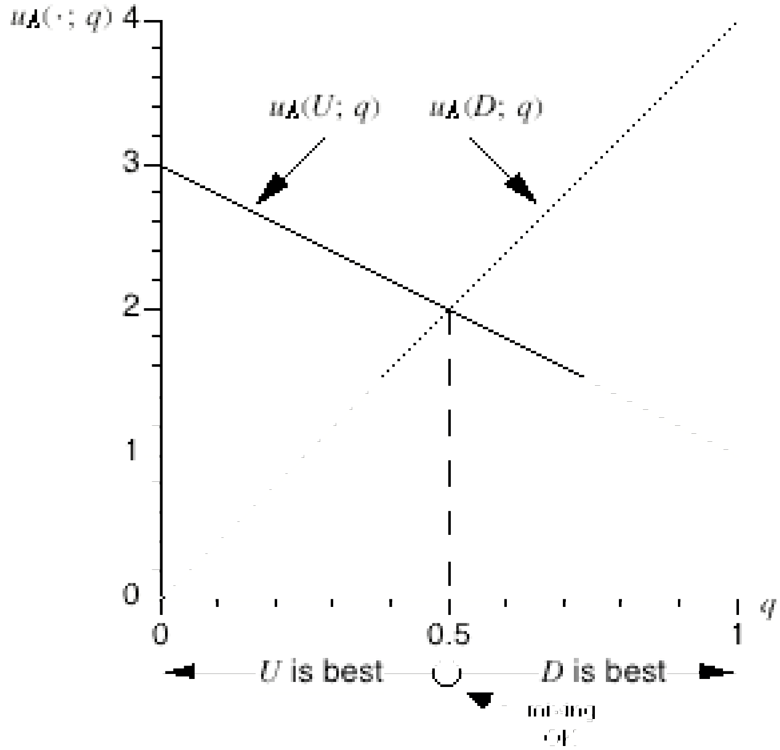

Figure 7: A's pure-strategy payoffs as a function of B's mixed strategy q.

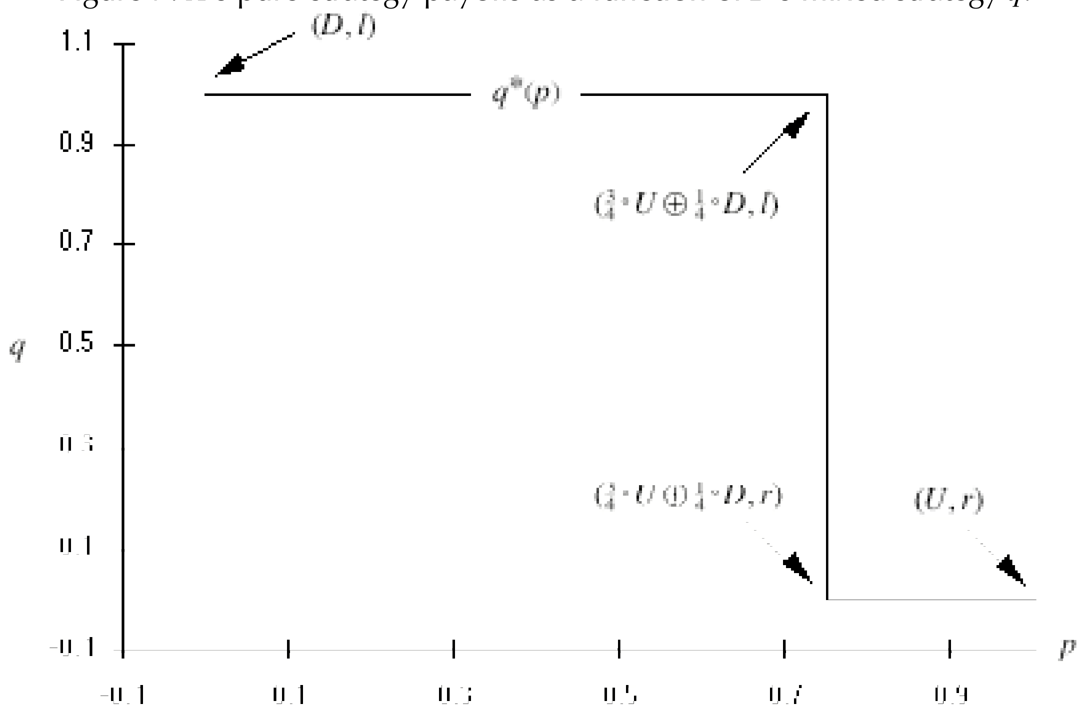

Figure 8: B's best-response correspondence for the game of Figure 5.

Using (7) and (14) we compute B's payoff to the mixed-strategy profile (p, q) to be

$$u_B(q;\, p) = (-4p+3)q + (p-1). \tag{15}$$

Evaluating this at B's pure-strategy choices we get

$$u_B(l;\, p) = 2 - 3p, \tag{16}$$

$$u_B(r;\, p) = p - 1. \tag{17}$$

These functions are plotted for $p \in [0, 1]$ in Figure 9. We again note that their intersection—in this case at $p^\dagger = {}^3\!/_4$—occurs at the opponent's mixed strategy which allows mixing by the player choosing a best response. The same reasoning we used above to graphically derive A's best-response correspondence works here, and we arrive at the same behavior rules we found analytically.

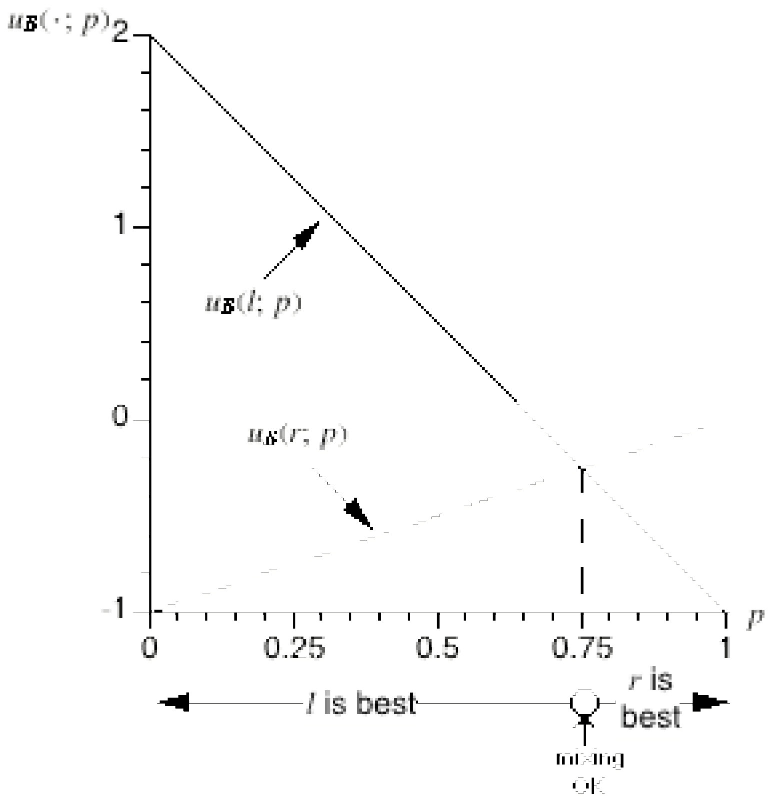

Figure 9: B′s pure-strategy payoffs as a function of A′s mixed strategy p.

The Nash set

Both A's and B's best-response correspondences are plotted together in Figure 10. We see that the intersection of the graphs of the two best-response correspondences contains exactly three points, each corresponding to a mixed-strategy profile (p, q): $(0, 1)$, $({}^3\!/_4, {}^1\!/_2)$, and $(1, 0)$. The first and last of these correspond to the two pure-strategy Nash equilibria we identified earlier. Note that the additional equilibrium we found is the strategy profile $(p^\dagger, q^\dagger)$. This strategy profile will in general be the only

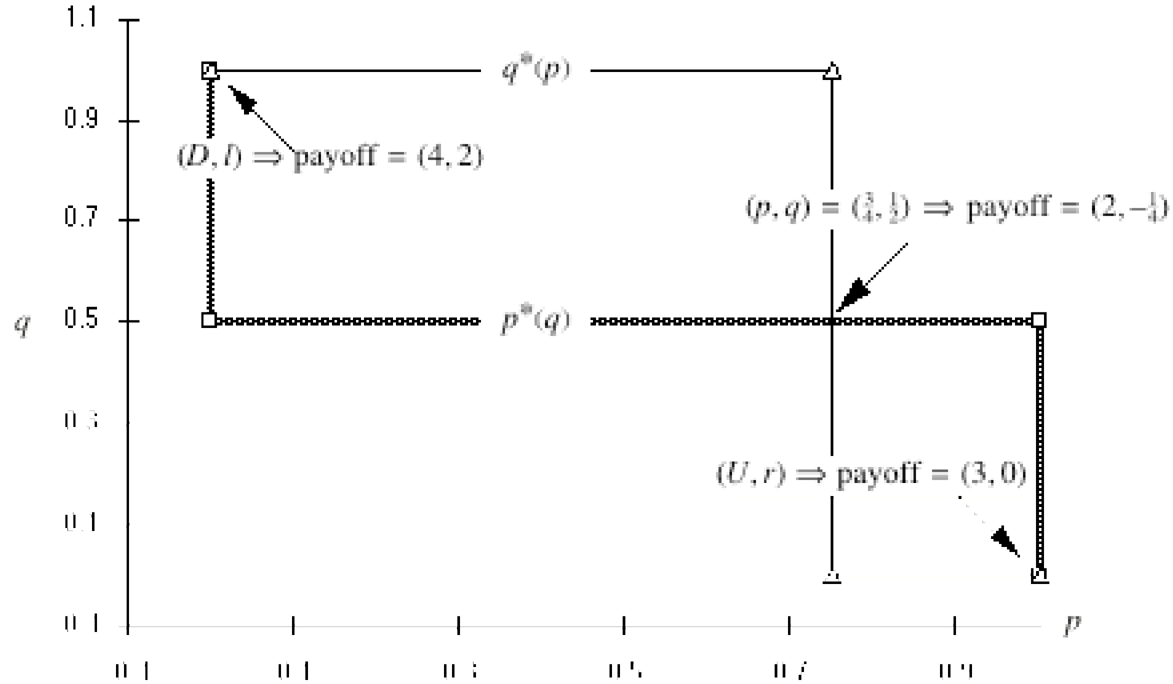

Figure 10: The players' best-response correspondences,
the Nash set, and equilibrium payoffs.

non-pure equilibrium strategy profile when $p^\dagger$ and $q^\dagger$ both lie in the interior of the unit interval.[1] Note as well that there is an odd number of Nash equilibria for this game, as is "almost always" the case. The payoff vectors for these equilibria are $(4,2)$, $(2,-\frac{1}{4})$, and $(3,0)$, respectively. [The mixed-strategy profile payoffs are computed using (11) and (15).] Note that the equilibrium payoffs are completely Pareto ranked.

A nongeneric example

We now consider the two-player normal form game in Figure 11. We immediately determine that the unique pure-strategy equilibrium is (D,r).

	l: q	r: $1-q$
U: p	1, 3	3, 1
D: $1-p$	4, 2	3, 3

Figure 11: A nongeneric two-player game.

To find A's best-response correspondence we use (5) to compute that

$$\delta(q) = -3q, \tag{18}$$

1 It is not possible when $p^\dagger, q^\dagger \in (0,1)$ that in equilibrium only one player mixes. Assume, say, that A mixes. We know that A will mix only when $q = q^\dagger \in (0,1)$. Therefore B is mixing as well.

which decreases in q and vanishes at $q^\dagger = 0$. Therefore A plays the pure strategy $p = 0$ against any $q \in (0, 1]$ and she is willing to mix against $q = q^\dagger = 0$. Therefore D is a weakly dominant strategy for A. Her best-response correspondence $p^*(q)$ is plotted in Figure 12.

To find B's best-response correspondence we use (8) and find that

$$\gamma(p) = \tfrac{3}{2}p - 1. \tag{19}$$

This function increases in p and vanishes at $p^\dagger = {}^2/_3$. Therefore B will play the pure strategy $q = 0$ against any $p \in [0, {}^2/_3)$, will play the pure strategy $q = 1$ against any $p \in ({}^2/_3, 1]$, and will be free to mix for $p = p^\dagger = {}^2/_3$. B's best-response correspondence $q^*(p)$ is also plotted in Figure 12.

Inspection of Figure 12 shows that the intersection of the graphs of A's and B's best-response correspondences is a line segment along which B plays $q = 0$ and A mixes with any probability p on $[0, {}^2/_3]$. We note that the unique pure-strategy Nash equilibrium we identified earlier is the left endpoint of this set. This example is nongeneric in that we have an infinity (in fact, a continuum) of equilibria: a situation which generically never happens.

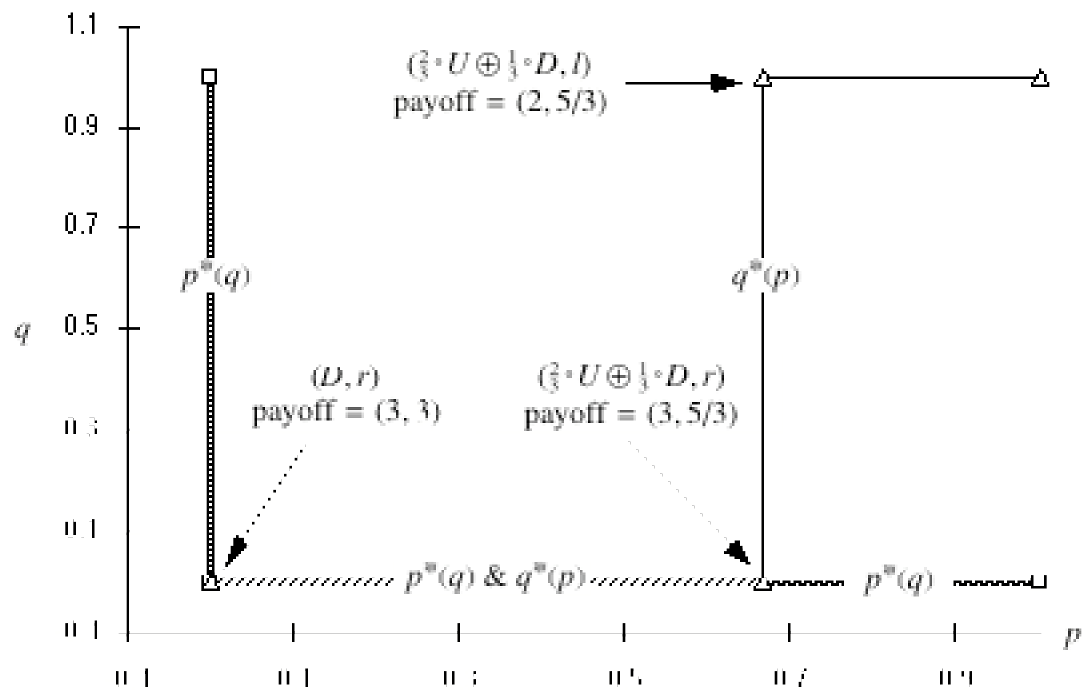

Figure 12: A game with a continuum of equilibria.

Before leaving this example we should also take note of the equilibrium payoffs. At the pure-strategy equilibrium each player gets 3. (See Figure 12.) Player A is indifferent to mixing between U and D, given that B is playing r. However, this mixing hurts B. At the right-hand endpoint of the Nash set, A still receives 3 but B's payoff, which has been decreasing linearly with A's mixing probability p, has declined to ${}^5/_3$. Note that, at the alternative strategy profile in which B plays l and A mixes with $p = {}^2/_3$, B would get the same payoff as playing r, but A would get only 2, rather than 3. If A mixed any more strongly toward U than $p = {}^{2}/_{3}$, B would defect to the alternative strategy l, giving A less than in this equilibrium. This is what determines the location of the right-hand endpoint of the Nash set. In this game the set of equilibria are Pareto ranked. Obviously, player B would prefer to coordinate on the pure-strategy equilibrium, and there is no reason A should disagree.